On the game Riccati equations arising in Hinfini control problems

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Abstract: In the state-space approach to H_∞ optimal control, feasibility of some closed-loop attenuation γ is characterized in terms of a pair of game Riccati equations depending on γ. This paper is concerned with the properties of these equations as γ varies. The most general problem is considered ($D_{11} \neq 0$) and a thorough analysis of the variations of the Riccati solutions provides insight into the behavior near the optimum and into the dependence on γ of the suboptimality conditions. In addition, concavity is established for a criterion which synthesizes the three conditions $X \geq 0$, $Y \geq 0$, and $\rho(XY) < \gamma^2$. As a result, a numerically reliable Newton scheme can be devised to compute the optimal γ.

Most presented results are extensions of earlier contributions. The main concern here is to provide a complete and synthetic overview as well as results and formulas tailored to the development of numerically-sound algorithms.

Sur les Équations de Riccati Associées au Problèmes de Contrôle H_∞

Résumé: Dans l'approche par variable d'état, la solvabilité d'un problème H_∞ sous-optimal est caractérisé en termes d'un couple d'équations de Riccati de type jeu et dépendant du gain γ. On étudie ici le comportement des solutions de ces équations lorsque γ varie. On considère le problème le plus général ($D_{11} \neq 0$). Cette analyse clarifie la façon dont les conditions de solvabilité évoluent avec γ. De plus, on montre que ces conditions peuvent être combinées en un unique critère qui dépend de façon concave de γ^{-1}. Cette propriété permet la mise en oeuvre de méthodes de Newton avec convergence quadratique pour le calcul du γ optimal.

1 Introduction

Many significant problems in linear system theory can be recast into the abstract framework of H_∞ optimal control. Well-known examples include model matching, disturbance attenuation, mixed sensitivity design, and robust stabilization in the face of uncertainty [5]. The general H_∞ optimal control problem can be stated as follows. Consider a plant G which maps exogenous inputs w and control inputs u to controlled outputs z and measured outputs y. That is,

$$\begin{pmatrix} z \\ y \end{pmatrix} = G(s)\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix}.$$

When G is closed by the output feedback law $u = K(s)y$, the closed-loop transfer function from w to z is given by the linear fractional map:

$$\mathcal{F}(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}. \tag{1.1}$$

The H_∞ optimal control problem consists of finding some real-rational, proper, and causal controller K_{opt} which internally stabilizes the plant while minimizing the norm $||\mathcal{F}(G,K)||_\infty$; that is,

$$||\mathcal{F}(G,K_{opt})||_\infty = \inf\{||\mathcal{F}(G,K)||_\infty : K \text{ internally stabilizes } G\}. \tag{1.2}$$

The infimum of all achievable gains is denoted γ_{opt}.

Although direct computation of γ_{opt} is a hard problem, the following suboptimal problem is relatively well understood and tractable:

$$\text{Given } \gamma > 0, \text{ does there exist an internally stabilizing } K \text{ such that } ||\mathcal{F}(G,K)||_\infty < \gamma\ ? \tag{1.3}$$

This paper is concerned with Doyle/Glover's state-space approach [4, 9] to solving this suboptimal problem. Over the past decade, this approach has emerged as the most direct and practical solution both on design and numerical grounds. It is now briefly summarized. To begin with, introduce a minimal realization of the plant G:

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(sI - A)^{-1}\begin{pmatrix} B_1 & B_2 \end{pmatrix}. \tag{1.4}$$

Here $A \in \mathbf{R}^{n \times n}$ and z, y, w, and u are vectors of size p_1, p_2, m_1, and m_2, respectively, with the assumption that $m_1 \geq p_2$ and $p_1 \geq m_2$. Accordingly, $D_{11} \in \mathbf{R}^{p_1 \times m_1}$, $D_{12} \in \mathbf{R}^{p_1 \times m_2}$, $D_{21} \in \mathbf{R}^{p_2 \times m_1}$, and $D_{22} \in \mathbf{R}^{p_2 \times m_2}$. Associate with this data the two Hamiltonian matrices:

$$H_\gamma = \begin{pmatrix} A & 0 \\ -C_1^TC_1 & -A^T \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ -C_1^TD_{11} & -C_1^TD_{12} \end{pmatrix}\begin{pmatrix} \gamma^2I - D_{11}^TD_{11} & -D_{11}^TD_{12} \\ -D_{12}^TD_{11} & -D_{12}^TD_{12} \end{pmatrix}^{-1}\begin{pmatrix} D_{11}^TC_1 & B_1^T \\ D_{12}^TC_1 & B_2^T \end{pmatrix}; \tag{1.5}$$

$$J_\gamma = \begin{pmatrix} A^T & 0 \\ -B_1B_1^T & -A \end{pmatrix} + \begin{pmatrix} C_1^T & C_2^T \\ -B_1D_{11}^T & -B_1D_{21}^T \end{pmatrix}\begin{pmatrix} \gamma^2I - D_{11}D_{11}^T & -D_{11}D_{21}^T \\ -D_{21}D_{11}^T & -D_{21}D_{21}^T \end{pmatrix}^{-1}\begin{pmatrix} D_{11}B_1^T & C_1 \\ D_{21}B_1^T & C_2 \end{pmatrix}. \tag{1.6}$$

Given a Hamiltonian matrix $H = \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}$ where R, Q are symmetric matrices, we denote by $\mathcal{X}_-(H)$ the stable invariant subspace of H. Associated with H is the algebraic Riccati equation $A^TX + XA + XRX + Q = 0$. Recall that this equation has a (unique) symmetric stabilizing solution if and only if H has no pure imaginary eigenvalue and $\mathcal{X}_-(H)$ is complementary to the subspace

$\mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix}$. When existing, such a solution is obtained as $X = QP^{-1}$ where $\begin{pmatrix} P \\ Q \end{pmatrix}$ is any basis of $\mathcal{X}_-(H) =$. In the sequel, $Ric(H)$ will refer to the stabilizing solution.

Throughout the paper, we call *general problem* (**GP**) the problem (1.3) with the standing assumptions:

(A1) (A, B_2, C_2) is stabilizable and detectable,

(A2) D_{12} has full column rank and D_{21} has full row rank.

(A3) $rank\begin{pmatrix} j\omega I - A & -B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m_2$ and $rank\begin{pmatrix} j\omega I - A & B_1 \\ -C_2 & D_{21} \end{pmatrix} = n + p_2$ for all $\omega \in \mathbf{R}$ or equivalently, G_{12} and G_{21} have no transmission zero on the imaginary axis [11],

(A4) $D_{22} = 0$.

While (A1) is necessary and sufficient for solvability of the GP for γ large enough, (A2)-(A3) are restrictive assumptions required for validity of Doyle/Glover's state-space results. Finally, (A4) amounts to a reparametrization of the controller set [9] and hence incurs no loss of generality.

Under assumptions (A1)-(A4), solvability of the GP with attenuation level γ is characterized by a set of conditions on the two Riccati equations associated with H_γ and J_γ [9].

Theorem 1.1 *With assumptions (A1)-(A4), there exists an internally stabilizing controller K such that $\|\mathcal{F}(G,K)\|_\infty < \gamma$ if and only if*

$$\gamma > \sigma_d := \max\left\{ \sigma_{max}\left((I - D_{12}(D_{12}^T D_{12})^{-1} D_{12}^T) D_{11}\right),\ \sigma_{max}\left(D_{11}(I - D_{21}^T (D_{21} D_{21}^T)^{-1} D_{21})\right)\right\} \tag{1.7}$$

and the following conditions hold:

(C1) *H_γ and J_γ have no eigenvalue on the imaginary axis,*

(C2) *neither $\mathcal{X}_-(H_\gamma)$ nor $\mathcal{X}_-(J_\gamma)$ intersects* $\mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix}$,

(C3) *$Ric(H_\gamma) =: X_\gamma \geq 0$ and $Ric(J_\gamma) =: Y_\gamma \geq 0$,*

(C4) $\rho(X_\gamma Y_\gamma) < \gamma^2$

Conditions (C1)-(C4) are often referred to as DGKF's conditions after the authors of [4]. Note that (1.7) ensures that the inverses in H_γ and J_γ are well-defined (see (3.2) together with the definition of $\hat{D}_{11}$). Note also that the Riccati equations associated with H_γ and J_γ have an indefinite quadratic term and are referred to as game Riccati equations (**GRE**). Finally, Theorem 1.1 characterizes γ_{opt} as the smallest $\gamma > 0$ for which the four conditions (C1)-(C4) are jointly satisfied. In turn, this suggests a straightforward bisection algorithm to compute γ_{opt}.

This paper examines the dependence on γ of conditions (C1)-(C4) and discusses its implications for the computation of the optimal gain γ_{opt}. To this purpose, the properties of the game Riccati equations associated with H_γ and J_γ are analyzed in detail. After simplifying the expressions of H_γ and J_γ by elementary reparametrizations, Section 3 characterizes those γ for which (C1) holds. In Section 4, the variations of the pseudoinverses of X_γ and Y_γ are shown to be smooth, monotonic and even concave wherever (C1) is satisfied. These results are obtained for the GP ($D_{11} \neq 0$) and allow a complete description of the variations of X_γ and Y_γ and of the behavior near γ_{opt} (Section 5). Finally, the computation of γ_{opt} is reformulated as a convex zero-crossing search problem which can be numerically solved by a Newton method (Section 6).

We conclude with a justification of our treatment of the GP instead of the simpler *Standard Problem* (**SP**) considered in [4]. Recall that the SP requires the additional assumptions

(A5) $D_{11} = 0$,

(A6) $D_{12}^T(D_{12}, C_1) = (I, 0)$ and $D_{21}(D_{21}^T, B_1^T) = (I, 0)$.

These assumptions have the advantage of notably simplifying the expressions of H_γ and J_γ. Moreover, such simplifications can be emulated for any GP via the loop-shifting techniques of [14]. Yet, these manipulations destroy the variational and structural properties of X_γ and Y_γ. Indeed, enforcing (A5) requires a γ-dependent transformation which alters the gradient and complicates its computation. In addition, the transformed problem is by no means a SP. In fact, the Riccati equations associated with GP's and SP's are structurally different as illustrated by Theorem 5.7 below. For theoretical and numerical reasons, it is therefore desirable to work in the GP framework.

2 Notation and Terminology

Given a square matrix M, a subspace $\mathcal{S}$ is said to be M-invariant if $M\mathcal{S} \subset \mathcal{S}$, and stable (antistable) M-invariant if moreover the restriction of M to $\mathcal{S}$ is stable (antistable). The following notation and definitions are used throughout the paper.

$\mathbf{C}_-, \mathbf{C}_0, \mathbf{C}_+$	open left-half plane, imaginary axis, and open right-half plane, respectively
$\mathrm{Ker}\, X, \mathrm{Im}\, X$	null and range spaces of a matrix X, respectively
$\mathcal{X}_-(M)$	negative M-invariant subspace
$\Lambda(X), \rho(X)$	spectrum and of spectral radius of a square matrix X, respectively.
$\sigma_{max}(X)$	largest singular value of the matrix X
$In(X)$	the inertia of $X = X^T$, that is, the triple (π, ν, ζ) where π, ν, ζ denote the number of positive, negative, and zero eigenvalues of X, respectively
$\mathcal{V}_0(C, A)$	A-invariant subspace associated with the stable, (C, A)-unobservable modes of A

3 Condition on the Hamiltonian Spectrum

This section examines which restriction is imposed on γ by condition (C1). It is shown that the region where H_γ has no eigenvalue on the imaginary axis is a half line $\gamma > \gamma_H$. A computable formula for the threshold γ_H is also derived. These results extend earlier work in [16] to the general case $D_{11} \neq 0$.

To simplify subsequent calculations, the expressions (1.5)-(1.6) of H_γ and J_γ are first condensed in terms of compound parameters. With $D_{12}^+ = (D_{12}^T D_{12})^{-1} D_{12}^T$ denoting the pseudoinverse of D_{12}, introduce the parameters:

$$\begin{aligned} \hat{A} := A - B_2 D_{12}^+ C_1; \quad \hat{B}_1 := B_1 - B_2 D_{12}^+ D_{11}; \quad \hat{B}_2 := B_2 (D_{12}^T D_{12})^{-1/2}; \\ \hat{C}_1 := (I - D_{12} D_{12}^+) C_1; \quad \hat{D}_{11} := (I - D_{12} D_{12}^+) D_{11}. \end{aligned} \tag{3.1}$$

By elementary algebra, H_γ can be rewritten:

$$H_\gamma = \begin{pmatrix} \hat{A} & -\hat{B}_2 \hat{B}_2^T \\ -\hat{C}_1^T \hat{C}_1 & -\hat{A}^T \end{pmatrix} + \begin{pmatrix} \hat{B}_1 \\ -\hat{C}_1^T \hat{D}_{11} \end{pmatrix} (\gamma^2 I - \hat{D}_{11}^T \hat{D}_{11})^{-1} (\hat{D}_{11}^T \hat{C}_1, \hat{B}_1^T). \tag{3.2}$$

Similarly, with $D_{21}^{+} := D_{21}^T(D_{21}D_{21}^T)^{-1}$ and

$$\tilde{A} := A - B_1 D_{21}^{+} C_2; \quad \tilde{C}_1 := C_1 - D_{11}D_{21}^{+}C_2; \quad \tilde{C}_2 := (D_{21}D_{21}^T)^{-1/2}C_2;$$
$$\tilde{B}_1 := B_1(I - D_{21}^{+}D_{21}); \quad \tilde{D}_{11} := D_{11}(I - D_{21}^{+}D_{21}), \tag{3.3}$$

J_γ can be simplified to

$$J_\gamma = \begin{pmatrix} \tilde{A}^T & -\tilde{C}_2^T\tilde{C}_2 \\ -\tilde{B}_1\tilde{B}_1^T & -\tilde{A} \end{pmatrix} + \begin{pmatrix} \tilde{C}_1^T \\ -\tilde{B}_1\tilde{D}_{11}^T \end{pmatrix}(\gamma^2 I - \tilde{D}_{11}\tilde{D}_{11}^T)^{-1}(\tilde{D}_{11}\tilde{B}_1^T, \tilde{C}_1). \tag{3.4}$$

Note that the underlying reparametrizations are independent of γ and transparent for analytical purposes since neither H_γ nor J_γ is altered. In addition, the stabilizability of (A, B_2) is equivalent to that of $(\hat{A}, \hat{B}_2)$ and similarly for the detectability of (C_2, A) and $(\tilde{C}_2, \tilde{A})$. Finally, observe that even though $D_{12}^T\hat{C}_1 = 0$ and $D_{21}\tilde{B}_1^T = 0$ and even when $D_{11} = 0$, the reduced expressions (3.2) and (3.4) cannot be associated with any particular SP since $\hat{A}$ and $\tilde{A}$ are distinct in general.

On their domain of existence, the GRE solutions $X_\gamma = Ric(H_\gamma)$ have the important property of sharing the same null space as γ varies [15]. In other words, the singular part of X_γ is independent of γ and coincides with the stable $(\hat{C}_1, \hat{A})$-unobservable subspace which we denote as $\mathcal{V}_0(\hat{C}_1, \hat{A})$. Of course a similar result holds for Y_γ. This structural property is instrumental to the subsequent variational analysis which involves forming the pseudoinverses of X_γ and Y_γ. Indeed, consider some orthogonal decomposition

$$U^T\hat{A}U = \begin{pmatrix} \hat{A}_{11} & 0 \\ \star & \hat{A}_{22} \end{pmatrix}; \quad \hat{C}_1 U = \begin{pmatrix} \hat{C}_{11} & 0 \end{pmatrix}; \quad U^T\hat{B}_1 = \begin{pmatrix} \hat{B}_{11} \\ \star \end{pmatrix}; \quad U^T\hat{B}_2 = \begin{pmatrix} \hat{B}_{21} \\ \star \end{pmatrix} \tag{3.5}$$

where $\hat{A}_{22}$ is stable, $(\hat{C}_{11}, \hat{A}_{11})$ has no stable unobservable mode, and $U = (U_1, U_2)$ is a γ-independent orthogonal matrix satisfying

$$\mathrm{Im}\ U_2 = \mathcal{V}_0(\hat{C}_1, \hat{A}); \qquad \mathrm{Im}\ U_1 = \mathcal{V}_0^{\perp}(\hat{C}_1, \hat{A}). \tag{3.6}$$

Then $\mathrm{Ker}\, X_\gamma = \mathrm{Im}\ U_2$ and $U^T X_\gamma U = \begin{pmatrix} \bar{X}_\gamma & 0 \\ 0 & 0 \end{pmatrix}$ where $\bar{X}_\gamma$ is nonsingular. Equivalently,

$$X_\gamma = U_1 \bar{X}_\gamma U_1^T \tag{3.7}$$

Hence $X_\gamma^+ = U_1\bar{X}_\gamma^{-1}U_1^T$ and since U_1 is independent of γ, the variations of X_γ or X_γ^+ are completely described by those of the invertible matrix $\bar{X}_\gamma$. Similarly, Y_γ can be written

$$Y_\gamma = V_1\bar{Y}_\gamma V_1^T \tag{3.8}$$

where V_1 is any orthogonal complement of $\mathcal{V}_0(\tilde{B}_1^T, \tilde{A}^T)$.

Remark 3.1 The (stable) unobservable modes of $(\hat{C}_1, \hat{A})$ are exactly the (stable) invariant zeros of $G_{12}(s)$, i.e., the complex numbers $s \in \mathbb{C}_-$ for which the system matrix $P(s) = \begin{pmatrix} sI - A & -B_2 \\ C_1 & D_{12} \end{pmatrix}$ looses rank. This follows from the identities

$$\begin{pmatrix} sI - \hat{A} & 0 \\ \hat{C}_1 & D_{12} \end{pmatrix} = \begin{pmatrix} I & -B_2D_{12}^+ \\ 0 & I \end{pmatrix} P(s) \begin{pmatrix} I & 0 \\ -D_{12}^+C_1 & I \end{pmatrix}$$

and $D_{12}^T\hat{C}_1 = 0$ whence $P(s)$ is (column) rank deficient iff. $\begin{pmatrix} sI - \hat{A} \\ \hat{C}_1 \end{pmatrix}$ is rank deficient, that is, iff. s is an unobservable mode of $(\hat{C}_1, \hat{A})$.

■

The first requirement (C1) for solvability of the GP is concerned with pure imaginary eigenvalues of H_γ nor J_γ. The following theorem characterizes the region where (C1) is satisfied.

Theorem 3.2 *Assume (A1)-(A4) and consider H_γ given by (1.5). Then there exists a finite real number $\gamma_H \geq \sigma_{max}(\hat{D}_{11})$ such that*

$$\Lambda(H_\gamma) \cap \mathbb{C}_0 = \emptyset \text{ if and only if } \gamma > \gamma_H. \tag{3.9}$$

Moreover, γ_H can be computed as (using the notation (3.5)):

$$\gamma_H = \|\hat{D}_{11} + \hat{C}_{11}(sI + \hat{A}_Z)^{-1}(\hat{B}_{11} + Z\hat{C}_{11}^T\hat{D}_{11})\|_\infty, \tag{3.10}$$

where Z is the unique symmetric nonnegative stabilizing solution of

$$-\hat{A}_{11}Z - Z\hat{A}_{11}^T - Z\hat{C}_{11}^T\hat{C}_{11}Z + \hat{B}_{21}\hat{B}_{21}^T = 0 \tag{3.11}$$

and $\hat{A}_Z := -\hat{A}_{11} - \hat{C}_{11}^T\hat{C}_{11}Z$ is the corresponding (stable) closed-loop matrix.

Proof: The proof is easily adapted from [16]. See Appendix A for details.

■

Hence part of the spectrum of H_γ migrates toward the imaginary axis as γ decreases. The first contact occurs for $\gamma = \gamma_H$ and those eigenvalues which then reach the imaginary axis remain on $\mathbb{C}_0$ for all $\sigma_{max}(\hat{D}_{11}) < \gamma \leq \gamma_H$. Meanwhile, other eigenvalues can still join them at some $\gamma < \gamma_H$. Note that (A3) is necessary and sufficient to ensure that $\gamma_H < +\infty$. Finally, if γ_J denotes the counterpart of γ_H for J_γ and γ^* is defined as

$$\gamma^* = \max(\gamma_H, \gamma_J), \tag{3.12}$$

it follows that (C1) holds if and only if $\gamma > \gamma^*$. Observe that $\gamma^* \geq \sigma_d$ of Theorem 1.1 since the matrices involved in the definition of σ_d are exactly $\hat{D}_{11}$ and $\tilde{D}_{11}$ while $\gamma_H \geq \sigma_{max}(\hat{D}_{11})$ from (3.10) and $\gamma_J \geq \sigma_{max}(\tilde{D}_{11})$ by duality.

4 Variational Properties

We now restrict our attention to the interval $(\gamma^*, +\infty)$ where (C1) is satisfied and examine the regularity and variations of the GRE stabilizing solutions X_γ and Y_γ. Direct characterization of these variations is rendered difficult by the discontinuities arising where the complementarity condition (C2) fails. Fortunately, this problem disappears when considering instead the pseudoinverses of X_γ and Y_γ. Introduced in [16] for the SP, this technique allows a simple and powerful description of the variations with γ. Indeed, the continuous extensions of X_γ^+ and Y_γ^+ turn out to be monotonic and concave functions of the parameter $\alpha = \gamma^{-2}$. These results are now extended to the GP framework ($D_{11} \neq 0$) with no major difficulty except perhaps for the concavity part. On duality grounds, only the case of X_γ is considered here.

Recall from Section 3 that whenever (C2) is satisfied, X_γ can be decomposed independently of γ as $X_\gamma = U_1\bar{X}_\gamma U_1^T$ where $\bar{X}_\gamma$ is nonsingular. Consequently the pseudoinverse X_γ^+ is obtained as $X_\gamma^+ = U_1\bar{X}_\gamma^{-1}U_1^T$ and its variations are entirely determined by those of $\bar{X}_\gamma^{-1}$. In the sequel, we therefore restrict our attention to $\bar{X}_\gamma^{-1}$ or more precisely to its continuous extension $W_X(\gamma)$ on $(\gamma^*, +\infty)$. This extension is easily defined in terms of the decomposition (3.5) and of the stable invariant subspace of the reduced Hamiltonian

$$\bar{H}_\gamma := \begin{pmatrix} \hat{A}_{11} & -\hat{B}_{21}\hat{B}_{21}^T \\ -\hat{C}_{11}^T\hat{C}_{11} & -\hat{A}_{11}^T \end{pmatrix} + \begin{pmatrix} \hat{B}_{11} \\ -\hat{C}_{11}^T\hat{D}_{11} \end{pmatrix} (\gamma^2 I - \hat{D}_{11}^T\hat{D}_{11})^{-1} \begin{pmatrix} \hat{D}_{11}^T\hat{C}_{11} & \hat{B}_{11}^T \end{pmatrix}. \tag{4.1}$$

Specifically, given any basis $\begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix}$ of $\mathcal{X}_-(\bar{H}_\gamma)$, $\bar{Q}$ is invertible since $(\hat{C}_{11}, -\hat{A}_{11})$ is detectable and $W_X(\gamma) := \bar{P}\,\bar{Q}^{-1}$ is well-defined for all $\gamma > \gamma_H$. With this definition, $W_X(\gamma) = \bar{X}_\gamma^{-1}$ whenever $\bar{X}_\gamma$ exists, that is, whenever (C2) is satisfied. Note that $W_X(\gamma)$ is the unique stabilizing solution of the GRE:

$$\begin{aligned} -(\hat{A}_{11} + \hat{B}_{11}R_\gamma^{-1}\hat{D}_{11}^T\hat{C}_{11})\,W_X(\gamma) - W_X(\gamma)\,(\hat{A}_{11} + \hat{B}_{11}R_\gamma^{-1}\hat{D}_{11}^T\hat{C}_{11})^T \\ -\gamma^2\,W_X(\gamma)\hat{C}_{11}^T S_\gamma^{-1}\hat{C}_{11}W_X(\gamma) + \hat{B}_{21}\hat{B}_{21}^T - \hat{B}_{11}R_\gamma^{-1}\hat{B}_{11}^T = 0 \end{aligned} \tag{4.2}$$

where $R_\gamma := \gamma^2 I - \hat{D}_{11}^T\hat{D}_{11} > 0$ and $S_\gamma := \gamma^2 I - \hat{D}_{11}\hat{D}_{11}^T > 0$, and that the corresponding (stable) closed-loop matrix is

$$A_W = -(\hat{A}_{11} + \hat{B}_{11}R_\gamma^{-1}\hat{D}_{11}^T\hat{C}_{11})^T - \gamma^2\hat{C}_{11}^T S_\gamma^{-1}\hat{C}_{11}W_X(\gamma). \tag{4.3}$$

Regularity and monotonicity of $W_X(\gamma)$ over the whole interval $(\gamma_H, +\infty)$ are established in the next theorem.

Theorem 4.1 *With assumptions (A1)-(A4) of Section 1, the matrix-valued function $W_X(\gamma)$ defined above has the following properties:*

1. *$W_X(\gamma)$ is infinitely differentiable on $(\gamma_H, +\infty)$.*
2. *$W_X(\gamma)$ is monotonically increasing with γ.*
3. *$\lim_{\gamma\to+\infty} W_X(\gamma)$ exists and is positive definite.*

Proof:
(1) For $\gamma > \gamma_H$, $W_X(\gamma)$ is the stabilizing solution of the GRE (4.2) whose parameters are infinitely differentiable functions of γ. By the Implicit Function Theorem applied to the stabilizing solution of algebraic Riccati equations (see, e.g., [6, 7, 13]), W_X is therefore infinitely differentiable.
(2) It is equivalent to (2) but more convenient to show that W_X is a decreasing function of the parameter $\alpha = \gamma^{-2}$. Introduce $R_\alpha := I - \alpha\hat{D}_{11}^T\hat{D}_{11} > 0$ and $S_\alpha := I - \alpha\hat{D}_{11}\hat{D}_{11}^T > 0$, and observe that $R_\gamma^{-1} = \alpha R_\alpha^{-1}$ and $\gamma^2 S_\gamma^{-1} = S_\alpha^{-1}$. Moreover,

$$\frac{d}{d\alpha}(R_\gamma^{-1}) = R_\alpha^{-2}; \quad \frac{d}{d\alpha}(\gamma^2 S_\gamma^{-1}) = S_\alpha^{-1}\hat{D}_{11}\hat{D}_{11}^T S_\alpha^{-1} = \hat{D}_{11}R_\alpha^{-2}\hat{D}_{11}^T \tag{4.4}$$

where the last identity follows from $S_\alpha^{-1}\hat{D}_{11} = \hat{D}_{11}R_\alpha^{-1}$. Differentiating (4.2) with respect to $\alpha = \gamma^{-2}$ then yields

$$A_W^T\frac{dW_X}{d\alpha} + \frac{dW_X}{d\alpha}A_W - \hat{B}_{11}R_\alpha^{-2}\hat{D}_{11}^T\hat{C}_{11}W_X - W_X\hat{C}_{11}^T\hat{D}_{11}R_\alpha^{-2}\hat{B}_{11}^T - W_X\hat{C}_{11}^T\hat{D}_{11}R_\alpha^{-2}\hat{D}_{11}^T\hat{C}_{11}W_X - \hat{B}_{11}R_\alpha^{-2}\hat{B}_{11}^T = 0$$

which can be rewritten more compactly as

$$A_W^T\frac{dW_X}{d\alpha} + \frac{dW_X}{d\alpha}A_W - (\hat{B}_{11} + W_X\hat{C}_{11}^T\hat{D}_{11})R_\alpha^{-2}(\hat{B}_{11} + W_X\hat{C}_{11}^T\hat{D}_{11})^T = 0. \tag{4.5}$$

Since A_W is stable and $R_\alpha > 0$, it follows from the Lyapunov Theorem that $\frac{dW_X}{d\alpha} \leq 0$, that is, W_X is a monotonically decreasing function of α and thus a monotonically increasing function of γ.
(3) As $\gamma \to +\infty$, $W_X(\gamma)$ tends to the nonnegative stabilizing solution of the LQG-type Riccati equation:

$$-\hat{A}_{11}W - W\hat{A}_{11}^T - W\hat{C}_{11}^T\hat{C}_{11}W + \hat{B}_{21}\hat{B}_{21}^T = 0.$$

Since $(\hat{A}, \hat{B}_2)$ and therefore $(\hat{A}_{11}, \hat{B}_{21})$ are stabilizable, this solution is nonsingular whence $W_X(\infty) > 0$.

■

In addition, W_X turns out to be a concave function of the parameter $\alpha = \gamma^{-2}$. This property is the foundation of the Newton algorithm proposed in Section 6 for the computation of the optimal gain γ_{opt}.

Theorem 4.2 *W_X is a monotonically decreasing concave function of the parameter $\alpha = \gamma^{-2}$.*

Proof: The monotonicity was established in Theorem 4.1. To establish concavity, we calculate the second derivative of W_X with respect to α. First observe from (4.3) and (4.4) that

$$\begin{aligned} \frac{dA_W}{d\alpha} &= -\hat{C}_{11}^T S_\alpha^{-1} \hat{C}_{11} \frac{dW_X}{d\alpha} - \hat{C}_{11}^T \hat{D}_{11} R_\alpha^{-2} \hat{B}_{11}^T - \hat{C}_{11}^T \hat{D}_{11} R_\alpha^{-2} \hat{D}_{11}^T \hat{C}_{11} W_X \\ &= -\hat{C}_{11}^T S_\alpha^{-1} \hat{C}_{11} \frac{dW_X}{d\alpha} - \hat{C}_{11}^T \hat{D}_{11} R_\alpha^{-2} F^T \end{aligned} \tag{4.6}$$

with $F := \hat{B}_{11} + W_X \hat{C}_{11}^T \hat{D}_{11}$. Then differentiate (4.5) with respect to α to obtain

$$\begin{aligned} & A_W^T \frac{d^2 W_X}{d\alpha^2} + \frac{d^2 W_X}{d\alpha^2} A_W + \frac{dA_W^T}{d\alpha}\frac{dW_X}{d\alpha} + \frac{dW_X}{d\alpha}\frac{dA_W}{d\alpha} \\ & \qquad - \frac{dW_X}{d\alpha} \hat{C}_{11}^T \hat{D}_{11} R_\alpha^{-2} F^T - F R_\alpha^{-2} \hat{D}_{11}^T \hat{C}_{11} \frac{dW_X}{d\alpha} - 2 F R_\alpha^{-3} \hat{D}_{11}^T \hat{D}_{11} F^T = 0, \end{aligned}$$

which combined to (4.6) yields

$$\begin{aligned} & A_W^T \ddot{W}_X + \ddot{W}_X A_W - \\ & \quad 2\left\{ \frac{dW_X}{d\alpha} \hat{C}_{11}^T S_\alpha^{-1} \hat{C}_{11} \frac{dW_X}{d\alpha} + \frac{dW_X}{d\alpha} \hat{C}_{11}^T \hat{D}_{11} R_\alpha^{-2} F^T + F R_\alpha^{-2} \hat{D}_{11}^T \hat{C}_{11} \frac{dW_X}{d\alpha} + F R_\alpha^{-3} \hat{D}_{11}^T \hat{D}_{11} F^T \right\} = 0. \end{aligned} \tag{4.7}$$

A little algebra shows that

$$R_\alpha^{-3} \hat{D}_{11}^T \hat{D}_{11} = R_\alpha^{-2} \hat{D}_{11}^T \hat{D}_{11} R_\alpha^{-1} = R_\alpha^{-2} \left\{ \hat{D}_{11}^T \hat{D}_{11} R_\alpha \right\} R_\alpha^{-2} = R_\alpha^{-2} \hat{D}_{11}^T S_\alpha \hat{D}_{11} R_\alpha^{-2}$$

so that the bracketed term in (4.7) can be factorized to obtain:

$$A_W^T \frac{d^2 W_X}{d\alpha^2} + \frac{d^2 W_X}{d\alpha^2} A_W - 2(S_\alpha^{-1/2} \hat{C}_{11} \frac{dW_X}{d\alpha} + S_\alpha^{1/2} \hat{D}_{11} R_\alpha^{-2} F^T)^T (S_\alpha^{-1/2} \hat{C}_{11} \frac{dW_X}{d\alpha} + S_\alpha^{1/2} \hat{D}_{11} R_\alpha^{-2} F^T) = 0. \tag{4.8}$$

The stability of A_W then clearly ensures $\frac{d^2 W_X}{d\alpha^2} \leq 0$, that is, the concavity of W_X as a function of $\alpha = \gamma^{-2}$.

■

Once established for $\alpha = \gamma^{-2}$, similar conclusions in terms of the parameters γ^2 and γ^{-1} easily follow from the differentiation formulas for the composition of functions.

Corollary 4.3 *W_X is decreasing and concave as a function of γ^{-1} and is increasing and concave as a function of γ^2.*

Proof: Omitted for brevity.

■

5 Variations with γ and Behavior near the optimum

This section gathers the results accumulated so far to give a complete description of the dependence on γ of conditions (C1)-(C4) of Section 1. The role played by these conditions in the determination of γ_{opt} is also addressed and a few examples demonstrate that any one of (C1)-(C4) can fail at γ_{opt}.

Firstly, the theorems of Section 4 are applied to characterizing the behavior of X_γ and Y_γ over the half line $(\gamma^*, +\infty)$. The following technical lemma is a useful preliminary.

Lemma 5.1 *There are at most n isolated points in $(\gamma_H, +\infty)$ where $W_X(\gamma)$ is singular.*

Proof: See Appendix B.

■

We can now proceed with a corollary of Theorem 4.1 which describes the variations of X_γ.

Corollary 5.2 *The GRE stabilizing solution $X_\gamma = Ric(H_\gamma)$ has the following properties on the interval $(\gamma_H, +\infty)$:*

1. *X_γ is defined for all $\gamma > \gamma_H$ but at most n points $\gamma_1 < \cdots < \gamma_k$ $(0 \leq k \leq n)$ where $\|X_\gamma\| = +\infty$. Moreover, $\gamma_1, \cdots, \gamma_k$ are exactly the points where $W_X(\gamma)$ is singular.*

2. *With the convention $\gamma_0 := \gamma_H$ and $\gamma_{k+1} := +\infty$, X_γ is monotonically decreasing and of constant inertia on each interval $(\gamma_i, \gamma_{i+1}), i = 0 : k$.*

3. *When γ traverses some point of discontinuity γ_i from γ_i^+ to γ_i^-, at least one positive eigenvalue of X_γ disappears at $+\infty$ and reappears at $-\infty$.*

Proof: This corollary is an immediate consequence of Theorem 4.1 and of the fact that $\bar{X}_\gamma = W_X^{-1}(\gamma)$ whenever (C2) holds. To be convinced that the inertia of X_γ is constant on each (γ_i, γ_{i+1}), just observe that $\bar{X}_\gamma$ is nonsingular and continuous on these intervals which prohibits any change of inertia.

■

Remarkably, sign changes in the eigenvalues of X_γ never result from zero-crossing. Instead, positive eigenvalues change sign by escaping to $+\infty$ and reappearing at $-\infty$. As γ decreases from $+\infty$ in particular, $\|X_\gamma\|$ necessarily blows up when (C3) is about to fail. Consequently, (C2) and (C3) are tied together. Also, high gains are to be expected near γ_{opt} if (C3) fails first (e.g., full-state feedback) or if (C4) fails near γ_k. These various observations are illustrated on a simple example.

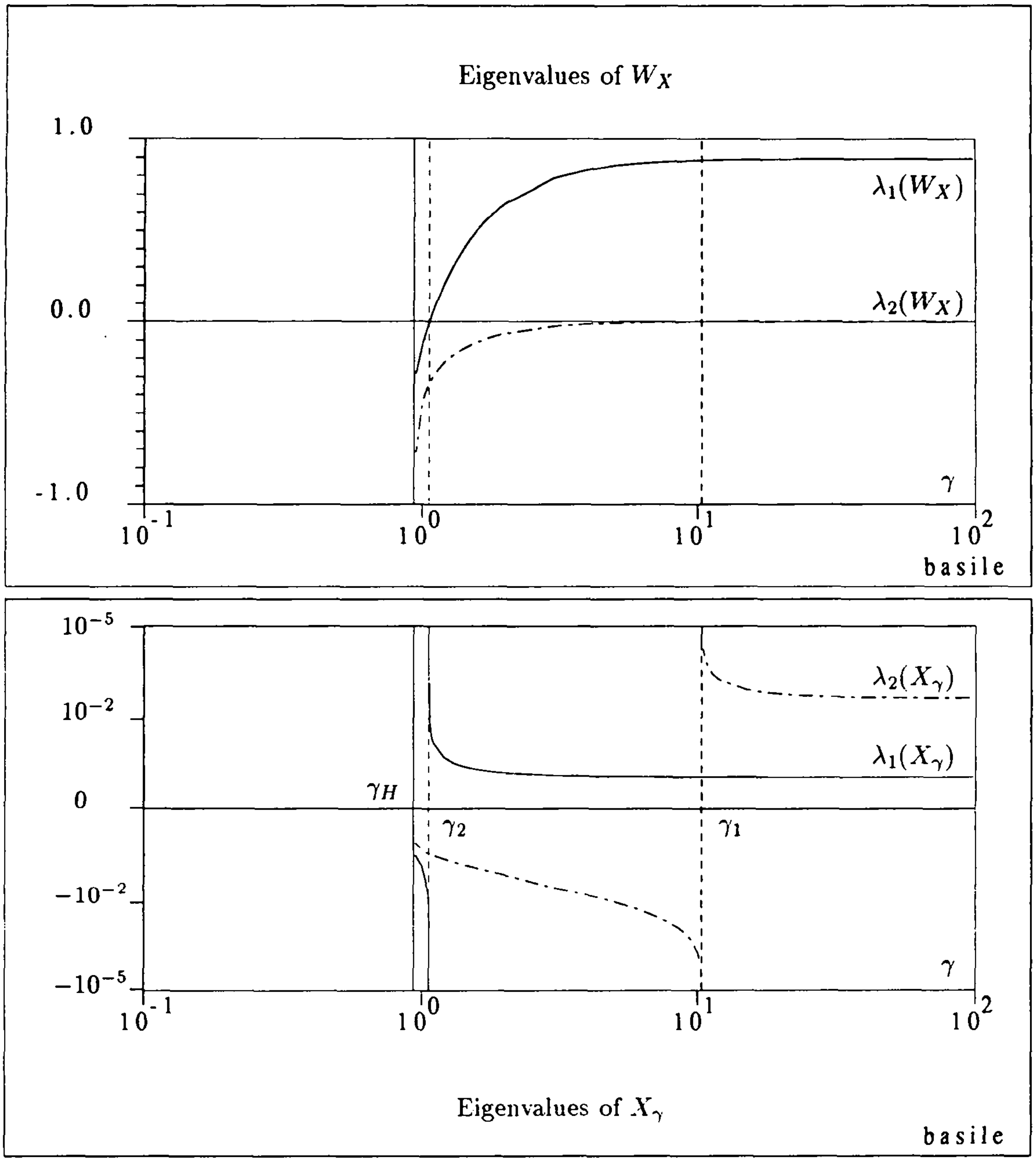
Example 5.3 Consider the GP with matrix data:

$$A = \begin{pmatrix} 10 & 2 \\ -1 & 8 \end{pmatrix}; \; B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \; B_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}; \; C_1 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}; \; D_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; D_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We focus on the variations of X_γ and of its inverse $W_X(\gamma)$ (here X_γ is nonsingular whenever defined). The eigenvalues of these two matrix-valued functions are plotted in Graph 5.4 for $\gamma > \gamma_H$ (here, $\gamma_H \approx 0.933$). The largest and smallest eigenvalues of W_X are denoted by $\lambda_1(W_X)$ and $\lambda_2(W_X)$, respectively, while $\lambda_1(X_\gamma)$ and $\lambda_2(X_\gamma)$ are taken as

$$\lambda_1(X_\gamma) = 1/\lambda_1(W_X); \quad \lambda_2(X_\gamma) = 1/\lambda_2(W_X).$$

Solid lines are used for the plots of $\lambda_1(W_X)$ and $\lambda_1(X_\gamma)$ and dashed lines for the plots of $\lambda_2(W_X)$ and $\lambda_2(X_\gamma)$. Note also the semilogarithmic scale for the eigenvalues of X_γ.


Eigenvalues of W_X
1.0
0.0
-1.0
$\lambda_1(W_X)$
$\lambda_2(W_X)$
γ
10^{-1}
10^{0}
10^{1}
10^{2}
basile
10^{-5}
10^{-2}
0
-10^{-2}
-10^{-5}
$\lambda_2(X_\gamma)$
$\lambda_1(X_\gamma)$
γ_H
γ_2
γ_1
γ
10^{-1}
10^{0}
10^{1}
10^{2}
Eigenvalues of X_γ
basile

Graph 5.4

Inspection of Graph 5.4 shows that X_γ has two points of discontinuity γ_1 and γ_2 in $(\gamma_H, +\infty)$ which correspond to the zero-crossings of $\lambda_1(W_X)$ and $\lambda_2(W_X)$. Moreover, the inertia of X_γ changes when traversing either one of these discontinuities so that $In(X_\gamma) = (2,0,0)$ for $\gamma > \gamma_1$, $In(X_\gamma) = (1,0,1)$ for $\gamma_2 < \gamma < \gamma_1$, and $In(X_\gamma) = (0,0,2)$ for $\gamma_H < \gamma < \gamma_2$. Finally, X_γ has a finite limit as $\gamma \to \gamma_H^+$.

■

Regarding the behavior near γ_{opt}, in most problems condition (C4) is the first to fail when approaching γ_{opt} from above. Yet, (C1) or (C2)/(C3) can sometimes fail first instead as demonstrated by the next two examples.

Example 5.5 This example is to illustrate that (C1) can fail first at γ_{opt}. Consider a plant $G(s)$ of realization (1.4) with $C_1 = I_2$, $D_{21} = 1$, $D_{22} = 0$, and

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad B_1 = B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad C_2^T = D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad D_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Clearly, assumption (A1) of Section 1 is satisfied since A is stable and (A2)-(A3) are trivially verified. After calculation of H_γ and J_γ via (1.5)-(1.6), elementary algebra shows that $\Lambda(H_\gamma)$ intersects $\mathbb{C}_0$ for $\gamma \le 1/\sqrt{2}$ while J_γ never has any eigenvalue on $\mathbb{C}_0$. Therefore $\gamma^* = 1/\sqrt{2}$. For all $\gamma \ge \gamma^*$, X_γ is obtained as the stabilizing solution of the GRE associated with H_γ:

$$X_\gamma = \begin{pmatrix} x^2(x+\frac{1}{2}) & -x^2 \\ -x^2 & x \end{pmatrix} \quad \text{where } x = \frac{1}{1+\sqrt{2-\gamma^{-2}}}. \tag{5.1}$$

Also, it is easily verified that $Y_\gamma = 0$ for all $\gamma > \gamma^*$. Hence, (C2)-(C4) reduce to the sole condition $X_\gamma \ge 0$ which is satisfied for all $\gamma \ge \gamma^*$ as it can be seen from (5.1). Consequently, $\gamma_{opt} = 1/\sqrt{2}$ and (C1) is the first condition to fail as γ decreases.

Example 5.6 This second example shows that (C3) can fail before (C4) in the GP context. That is, $||X_\gamma||$ or $||Y_\gamma||$ can become unbounded while $\rho(X_\gamma Y_\gamma)$ remains smaller than γ^2. Consider this time the plant $G(s)$ defined by the parameters $B_1 = C_1 = I_2$, $D_{22} = 0$, and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad B_2 = C_2^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad D_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad D_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad D_{21} = (1 \;\; 0).$$

It is easily verified that (A, B_2, C_2) is stabilizable and detectable. Via the reparametrization (3.1) and (3.3) we find $U_1 = V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_1 = U_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\tilde{H}_\gamma = \tilde{J}_\gamma = \begin{pmatrix} 0 & -1+\gamma^{-2} \\ -1 & 0 \end{pmatrix}$. Hence $\gamma^* = 1$ and for $\gamma > 1$, $W_X(\gamma) = W_Y(\gamma) = \sqrt{1-\gamma^{-2}}$ and

$$X_\gamma = U_1 W_X^{-1} U_1^T = \begin{pmatrix} \frac{1}{\sqrt{1-\gamma^{-2}}} & 0 \\ 0 & 0 \end{pmatrix}; \qquad Y_\gamma = V_1 W_X^{-1} V_1^T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\gamma^{-2}}} \end{pmatrix}$$

Consequently, $\rho(X_\gamma Y_\gamma) \equiv 0$ on $(\gamma^*, +\infty)$ while $\gamma_{opt} = 1$ since (C3) holds iff. $\gamma > 1$. Thus, (C3) fails at γ_{opt} while (C4) is trivially satisfied everywhere.

■

Incidentally, (C2)/(C3) never fail before (C4) in the restrictive framework of the SP. This fact critically relies on the particular duality relationship between H_γ and J_γ in the SP context.

Theorem 5.7 *With the Standard Problem assumptions, $\rho(X_\gamma Y_\gamma)$ cannot remain bounded when $||X_\gamma||$ or $||Y_\gamma||$ become unbounded. Consequently, (C2)/(C3) cannot fail before (C4) as γ decreases from $+\infty$.*

Proof: See Appendix C.

■

6 Computation of γ_{opt} by a Newton Method

The concavity results of Section 5 readily suggest a Newton scheme to locate the first failure of condition (C3). To extend such a scheme to the computation of γ_{opt}, (C4) must also be recast as a concave constraint. Such a reformulation is attempted in [16] but the concave criterion proposed therein to replace (C4) has two major drawbacks: it applies only in the region where (C3) is satisfied, and it mixes X_γ, Y_γ, and their pseudoinverses. In this section, we introduce an alternative criterion which combines (C2)-(C4) into a single concave constraint. First introduced in [10], this criterion applies to the whole region $\gamma > \gamma^*$ and solely involves the pseudoinverses of X_γ and Y_γ.

Theorem 6.1 *Let $U = (U_1, U_2)$ and $V = (V_1, V_2)$ be orthogonal transformations of $\mathbf{R}^{n\times n}$ satisfying*

$$\text{Im } U_2 = \mathcal{V}_0(\hat{C}_1, \hat{A}); \qquad \text{Im } V_2 = \mathcal{V}_0(\tilde{B}_1^T, \tilde{A}^T). \tag{6.1}$$

With γ^* given by (3.12), define for $\gamma > \gamma^*$ the matrix-valued function Z by

$$Z(\gamma^{-1}) := \begin{pmatrix} W_X(\gamma) & \gamma^{-1}U_1^T V_1 \\ \gamma^{-1}V_1^T U_1 & W_Y(\gamma) \end{pmatrix} \tag{6.2}$$

where W_X denotes the continuous extension of $\bar{X}_\gamma^{-1}$ in (3.7) and W_Y that of $\bar{Y}_\gamma^{-1}$ in (3.8). Then Z is a concave function of the parameter γ^{-1} on the half line $\gamma > \gamma^*$. Moreover, there is equivalence between:

(1) *There exists a stabilizing controller $K(s)$ such that $\|\mathcal{F}(K,G)\|_\infty < \gamma$;*

(2) *H_γ and J_γ have no eigenvalue on $\mathbf{C}_0$ and $Z(\gamma^{-1})$ is positive definite.*

Proof: Both W_X, W_Y and therefore $Z(\gamma^{-1})$ are well-defined for $\gamma > \gamma^*$ and the concavity follows from $\frac{d^2 Z}{d(\gamma^{-1})^2} = \begin{pmatrix} \frac{d^2 W_X}{d(\gamma^{-1})^2} & 0 \\ 0 & \frac{d^2 W_Y}{d(\gamma^{-1})^2} \end{pmatrix}$ together with Corollary 4.3. As of the equivalence of (1) and (2), it can be found in [10].

■

Hence, the condition $Z(\gamma^{-1}) > 0$ emerges as a more natural and compact characterization of suboptimal γ's. Moreover, it synthesizes (C2)-(C4) into a single concave constraint so that the computation of γ_{opt} reduces to finding the zero-crossing of a concave function. Given some initial guess in the interval (γ^*, γ_{opt}), this problem can be solved by a Newton method with guaranteed quadratic convergence (see Theorem 6.2 below). Details of implementation appear in [8]. The only delicate issue is the initialization of the algorithm. Fortunately, Theorem 3.2 provides explicit formulas for γ^*. To estimate γ^* indeed, it suffices to solve the associated H_2 problem ($\gamma = +\infty$) and to compute the L_∞ norm of two transfer functions, a task for which quadratically convergent algorithms are also available [3].

This section is concluded with a theorem which formalizes the Newton method for the computation of γ_{opt} and gives a computable characterization of the Newton step.

Theorem 6.2 *Introduce the new parameter $\theta := \gamma^{-1}$, let*

$$\theta_{opt} := \gamma_{opt}^{-1}; \qquad \theta^* := (\gamma^*)^{-1}, \tag{6.3}$$

and consider the function $s(\theta)$ defined for $\theta \in (\theta_{opt}, \theta^*)$ by:

$$s(\theta) := \inf\left\{ s > 0 \ : \ Z(\theta) - s\,\frac{dZ}{d\theta}(\theta) \geq 0 \right\}. \tag{6.4}$$

Then, given any $\theta_0 \in (\theta_{opt}, \theta^*)$, the sequence

$$\theta_{k+1} = \theta_k - s(\theta_k); \quad k = 0, 1, \cdots \tag{6.5}$$

is monotonic and quadratically converges to $\theta_{opt} = \gamma_{opt}^{-1}$ from above. Moreover, the Newton step $s(\theta)$ can be computed as the largest eigenvalue in $(0, \theta)$ of the pencil $Z(\theta) - \lambda \frac{dZ}{d\theta}(\theta)$.

Proof: First observe that the set S involved in the definition of $s(\theta)$ is nonempty since, by concavity, $Z(\theta - s) \leq Z(\theta) - s\,\frac{dZ}{d\theta}(\theta)$ for all $s > 0$ whence

$$0 \leq Z(\theta_{opt}) \leq Z(\theta) - (\theta - \theta_{opt})\,\frac{dZ}{d\theta}(\theta) \tag{6.6}$$

and $0 < \theta - \theta_{opt} \in S$. Consequently, s is well-defined for all $\theta \in (\theta_{opt}, \theta^*)$ and maps (θ_{opt}, θ^*) into itself since from (6.6), $s(\theta) \leq \theta - \theta_{opt}$ and therefore $\theta_{opt} \leq \theta - s(\theta) \leq \theta \leq \theta^*$.

To show quadratic convergence of θ_k to θ_{opt}, recall that $Z(\theta_{opt})$ is singular and consider a unit vector w such that $Z(\theta_{opt})w = 0$. The function $f_w(\theta) := w^T Z(\theta) w$ inherits the concavity of $Z(\theta)$ and also its positivity for $\theta \in (0, \theta_{opt})$ (cf. Theorem 6.1). Hence, $f_w(\theta_{opt}) = 0$ implies that $f_w(\theta) \leq 0$ and $\frac{df_w}{d\theta}(\theta) < 0$ for all θ in $[\theta_{opt}, \theta^*)$. By a well-known result of convex analysis, the Newton iterations

$$\phi_0 = \theta_0; \quad \phi_{k+1} = \phi_k - \frac{f_w}{df_w/d\theta}(\phi_k) \tag{6.7}$$

are then quadratically convergent (from above) to the zero-crossing of f_w, that is, θ_{opt}. Now, remark from the definition of $s(\theta)$ that $w^T(Z(\theta) - s(\theta)\,\frac{dZ}{d\theta}(\theta))w \geq 0$ for all $\theta \in (\theta_{opt}, \theta^*)$, that is, $f_w(\theta) - s(\theta)\frac{df_w}{d\theta}(\theta) \geq 0$. It follows that $s(\theta) \geq \frac{f_w}{df_w/d\theta}(\theta)$ since $\frac{df_w}{d\theta}(\theta) < 0$. Consequently, the sequence (6.5) decreases faster than (6.7) and hence quadratically converges to θ_{opt} as well.

Finally, observe from (6.4) that $Z(\theta) - s(\theta)\frac{dZ}{d\theta}(\theta)$ is singular whence $s(\theta)$ is a generalized eigenvalue of the pencil $Z(\theta) - \lambda\frac{dZ}{d\theta}(\theta)$. Suppose this pencil has a larger eigenvalue σ in $(0, \theta)$. Then $u^T Z(\theta) u - \sigma\, u^T \frac{dZ}{d\theta}(\theta) u = 0$ for some vector $u \neq 0$. To obtain a contradiction, distinguish three cases depending on the sign of $g_u = u^T \frac{dZ}{d\theta}(\theta) u$:

- if $g_u > 0$, then by the concavity of Z,

  $$u^T Z(0) u \leq u^T Z(\theta) u - \theta\, u^T \frac{dZ}{d\theta}(\theta) u = -(\theta - \sigma) g_u < 0$$

  which contradicts the positivity of $Z(0)$;

- if $g_u = 0$, then $u^T Z(\tau) u \leq 0$ for all τ from the concavity of Z which again contradicts the positivity of $Z(0)$;

- if finally $g_u < 0$, then from the assumption $s(\theta) < \sigma$,

  $$u^T Z(\theta) u - s(\theta)\, u^T \frac{dZ}{d\theta} u = (\sigma - s(\theta)) g_u < 0$$

  which contradicts the definition of $s(\theta)$.

■

7 Conclusion

The dependence on γ of the solutions to the H_∞ control problem has been precisely characterized. As a result, insight was gained into the structure, singularities, inertia, and variations of these solutions, and into their behavior near the optimum. This information is valuable for numerically stable testing of the suboptimality conditions and for the computation of the optimal γ. In particular, the concavity properties established in Sections 4 and 6 allow the design of Newton algorithms which quadratically converge to γ_{opt}.

Appendix A

Proof of Theorem 3.2:

Using (3.5), it is easily verified that $\Lambda(H_\gamma)\cap\mathbb{C}_0 = \Lambda(\bar{H}_\gamma)\cap\mathbb{C}_0$ with $\bar{H}_\gamma$ as in (4.1). Now, $(\hat{A}_{11},\hat{B}_{21})$ inherits the stabilizability of $(\hat{A},\hat{B}_2)$ and $(\hat{C}_{11},-\hat{A}_{11})$ is detectable from (3.5) and (A3). Consequently [12], (3.11) has a unique symmetric nonnegative definite stabilizing solution Z. Now, it is easily verified that

$$-\begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix}\bar{H}_\gamma\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} \tilde{A}_Z - FR_\gamma^{-1}\hat{D}_{11}^T\hat{C}_{11} & -FR_\gamma^{-1}F^T \\ \gamma^2\hat{C}_{11}^TS_\gamma^{-1}\hat{C}_{11} & -\tilde{A}_Z^T + \hat{C}_{11}^T\hat{D}_{11}R_\gamma^{-1}F^T \end{pmatrix} := H_Z(\gamma)$$

where $F := \hat{B}_{11} + Z\hat{C}_{11}^T\hat{D}_{11}$. This last identity shows that $-\bar{H}_\gamma$ and $H_Z(\gamma)$ share the same spectrum. Moreover, from the Bounded Real Lemma [1, 2] we know that

$$\Lambda(H_Z(\gamma))\cap\mathbb{C}_0 = \emptyset \quad \text{iff} \quad \|\hat{D}_{11} + \hat{C}_{11}(sI - \tilde{A}_Z)^{-1}F\|_\infty < \gamma.$$

Consequenly, $\bar{H}_\gamma$ or equivalently H_γ has no eigenvalue on $\mathbb{C}_0$ if and only if $\gamma > \gamma_H := \|\hat{D}_{11} + \hat{C}_{11}(sI - \tilde{A}_Z)^{-1}F\|_\infty$. Note that $\gamma_H < +\infty$ since $\tilde{A}_Z$ is stable.

Appendix B

Proof of Lemma 5.1:

From the monotonicity of W_X, it suffices to show that the singularities of W_X are isolated. To this purpose, suppose that $W_{X,0} := W_X(\gamma_0)$ is singular and let $\dot{W}_{X,0} := \frac{dW_X}{d\alpha}(\gamma_0)$ denote its derivative. To prove that γ_0 is isolated, it is sufficient to show that $\operatorname{Ker} W_{X,0} \cap \operatorname{Ker}\dot{W}_{X,0} = \{\vec{0}\}$. This is established by contradiction.

Assume this intersection is nontrivial and spanned by the columns of a full-rank matrix L. Then $W_{X,0}L = 0$ together with (4.2) provides $L^T(\hat{B}_{21}\hat{B}_{21}^T - \hat{B}_{11}R_\gamma^{-1}\hat{B}_{11}^T)L = 0$, and $\dot{W}_{X,0}L = 0$ together with (4.5) provides $L^T\hat{B}_{11}R_\alpha^{-2}\hat{B}_{11}^TL = 0$. In turn, these two identities yield $\hat{B}_{11}^TL = \hat{B}_{21}^TL = 0$ and postmultiplying (4.2) and (4.5) by L then provides $\dot{W}_{X,0}A_WL = -\dot{W}_{X,0}\hat{A}_{11}^TL = 0$ and $W_{X,0}\hat{A}_{11}^TL = 0$, respectively. Consequently, $\operatorname{Im} L = \operatorname{Ker} W_{X,0}\cap\operatorname{Ker}\dot{W}_{X,0}$ is $\hat{A}_{11}^T$-invariant. Now, $\begin{pmatrix} W_{X,0} \\ I \end{pmatrix}$ spans $\mathcal{X}_-(\bar{H}_\gamma)$ whence $\begin{pmatrix} W_{X,0}L \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ L \end{pmatrix}$ is a subspace of $\mathcal{X}_-(\bar{H}_\gamma)$. Observing that

$$\bar{H}_\gamma\begin{pmatrix} 0 \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{A}_{11}^TL \end{pmatrix},$$

it follows that $\operatorname{Im} L$ is antistable $\hat{A}_{11}^T$-invariant. Together with $\hat{B}_{21}^TL = 0$, this contradicts the stabilizability of $(\hat{A}_{11},\hat{B}_{21})$ which is inherited from that of $(\hat{A},\hat{B}_2)$ or equivalently (A,B_2).

Appendix C

Proof of Theorem 5.7:

The proof is by contradiction. Assume for instance that X_γ is unbounded at γ_{opt}, that is, $W_{X,o} = W_X(\gamma_{opt})$ is singular. Introduce a matrix L whose columns form a basis of $\operatorname{Ker} W_{X,o}$. Since $Z(\gamma_{opt}^{-1}) \geq 0$, we have for all matrix M of compatible dimensions:

$$0 \leq (L^T \;\; M^T) Z(\gamma_{opt}^{-1}) \begin{pmatrix} L \\ M \end{pmatrix} = M^T W_{Y,o} M + \gamma_{opt}^{-1}(L^T U_1^T V_1 M + M^T V_1^T U_1 L).$$

This requires $V_1^T U_1 L = 0$, or equivalently $\operatorname{Im} U_1 L \subset \operatorname{Im} V_2 = \mathcal{V}_0(\tilde{B}_1^T, \tilde{A}^T)$. Observing that $(\tilde{A}^T, \tilde{B}_1^T) = (A^T, B_1^T)$ in the SP context, it follows that $\operatorname{Im} U_1 L$ is A^T-invariant and that $B_1^T U_1 L = 0$. Now, reporting $W_{X,o} L = 0$ in (4.2) provides $L^T(\hat{B}_{21}\hat{B}_{21}^T - \gamma^{-2}\hat{B}_{11}\hat{B}_{11}^T)L = 0$, or equivalently (using (3.5)):

$$0 = (U_1 L)^T(\hat{B}_2\hat{B}_2^T - \gamma^{-2}\hat{B}_1\hat{B}_1^T)(U_1 L) = (U_1 L)^T(B_2 B_2^T - \gamma^{-2}B_1 B_1^T)(U_1 L). \qquad \text{(C.1)}$$

This together with $B_1^T U_1 L = 0$ imposes $B_2^T U_1 L = 0$. A contradiction to the stabilizability of (A, B_2) can then be derived by an argument similar to that in the proof of Lemma 5.1.

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